

MAS212 Linear Algebra I
MID-TERM TEST

Date: 9 November 2007 *Time:* 11.10–11.50

Complete the following information:

Name	
Student Number (9 digit code)	

The duration of the test is **40 minutes**. Answer **all** 10 questions. Each question is worth 1 mark. Only the final answer to a question will be marked, so indicate this answer clearly. Calculators are **not** allowed.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets, but this will not be looked at.

Total Marks	
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1. Let

$$A = \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -5 & -1 \end{pmatrix}.$$

For each of the following matrix products, either evaluate it or state that it does not exist: AB , BA .

Answer. BA does not exist, and

$$AB = \begin{pmatrix} 4 & 1 & 3 \\ 5 & -29 & -10 \end{pmatrix}.$$

□

2. Exactly which of the following matrices (if any) are in reduced row echelon form:

$$(a) \quad \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (b) \quad \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Answer. Only (a).

□

3. Determine the solution set of the following linear system:

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & + & 2x_3 & - & x_4 & = & 2 \\ -x_1 & + & 2x_2 & - & 3x_3 & + & 4x_4 & = & 1 \\ & & 5x_2 & - & x_3 & + & 3x_4 & = & 4 \end{array}$$

Answer. Bring augmented matrix to row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 2 \\ -1 & 2 & -3 & 4 & 1 \\ 0 & 5 & -1 & 3 & 4 \end{array} \right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 2 \\ 0 & 5 & -1 & 3 & 3 \\ 0 & 5 & -1 & 3 & 4 \end{array} \right) \xrightarrow{R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 3 & 2 & -1 & 2 \\ 0 & 5 & -1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

The last line shows that the system is inconsistent, so the solution set is empty. $\square$

4. Find the inverse of the following matrix using Gauss-Jordan inversion:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}.$$

Answer. Gauss-Jordan inversion gives

$$\begin{array}{c} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -2 & 1 & -6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ \xrightarrow{R_1 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right). \end{array}$$

so

$$A^{-1} = \begin{pmatrix} 1 & 0 & -3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

5. Let

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 1 & -1 \\ 7 & 9 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & -1 \\ -1 & 1 & -1 \\ 7 & 9 & 2 \end{pmatrix}.$$

Find an elementary matrix $E \in \mathbb{R}^{3 \times 3}$ such that $EA = B$.

Answer. Since B is obtained from A by adding two times rows 2 to row 1,

$$E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

6. Let A and B be 4×4 matrices. Express $\det(B)$ in terms of $\det(A)$ in the following three cases:

- (a) B is obtained from A by first swapping row 1 and row 4, and then swapping row 2 and row 3;
- (b) B is obtained from A by adding 2 times row 3 to row 1;
- (c) B is obtained from A by multiplying row 2 by 5.

Answer. By Theorem 3.11:

- (a) $\det(B) = (-1)(-1)\det(A) = \det(A)$;
- (b) $\det(B) = \det(A)$;
- (c) $\det(B) = 5\det(A)$.

□

7. Let

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 7 & -5 & 3 & -1 \\ 4 & 9 & 5 & 0 \\ 1 & 8 & 2 & 0 \end{pmatrix}$$

Decide whether A is invertible. If it is, calculate the $(1, 4)$ -entry of A^{-1} .

Answer.

$$\det(A) = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 7 & -5 & 3 & -1 \\ 4 & 9 & 5 & 0 \\ 1 & 8 & 2 & 0 \end{vmatrix} = (-2) \begin{vmatrix} 7 & 3 & -1 \\ 4 & 5 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (-2)(-1) \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = (-2)(-1) \cdot 3 = 6 \neq 0,$$

so A is invertible. To calculate the $(4, 1)$ -entry of A^{-1} we calculate the $(4, 1)$ -cofactor C_{41} of A :

$$C_{41} = (-1)^{4+1} \begin{vmatrix} 2 & 0 & 0 \\ -5 & 3 & -1 \\ 9 & 5 & 0 \end{vmatrix} = (-1) \cdot 2 \cdot \begin{vmatrix} 3 & -1 \\ 5 & 0 \end{vmatrix} = (-1) \cdot 2 \cdot 5 = -10.$$

Hence, the $(1, 4)$ -entry of A^{-1} is

$$\frac{C_{41}}{\det(A)} = \frac{-10}{6} = -\frac{5}{3}.$$

□

8. Calculate $\det(A^5)$ where

$$A = \begin{pmatrix} 1 & -2 & 3 & 7 & 2 \\ 0 & 1 & 8 & -5 & 3 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Answer. The matrix A is upper triangular, so its determinant equals the product of its diagonal entries. Thus

$$\det(A) = 1 \cdot 1 \cdot 2 \cdot (-1) \cdot 1 = -2.$$

Thus

$$\det(A^5) = \det(A)^5 = (-2)^5 = -32.$$

□

9. Exactly which of the following statements (if any) are true:

- (a) $\{ A \in \mathbb{R}^{n \times n} \mid A \text{ is invertible} \}$ is a subspace of $\mathbb{R}^{n \times n}$.
- (b) If $A, B \in \mathbb{R}^{n \times n}$, then $(A + B)^2 = A^2 + 2AB + B^2$.
- (c) If $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x}$ is a non-zero vector in $\mathbb{R}^n$ with $A\mathbf{x} = \mathbf{0}$ then A is singular;
- (d) if $A, B \in \mathbb{R}^{n \times n}$ and $\det(A) = \det(B)$, then $A = B$.

Answer.

- (a) False. If it was a subspace, it should contain the zero matrix. However, the zero matrix is not invertible.
- (b) False. The statement holds if and only if A and B commute. This, however, is not true in general.
- (c) True by the Invertible Matrix Theorem.
- (d) False. Counterexample: let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\det(A) = \det(B) = 0$, but $A \neq B$.

□

10. Let $A \in \mathbb{R}^{n \times n}$ be invertible and suppose that $I - 3A + A^2 = O$. Express A^{-1} as a linear combination of I and A .

Answer. Since A is invertible, we can multiply both sides of

$$I - 3A + A^2 = O$$

from the left (or the right) by A^{-1} , which gives

$$A^{-1}I - A^{-1}(3A) + A^{-1}A^2 = A^{-1}O.$$

Thus

$$A^{-1} - 3I + A = O,$$

so

$$A^{-1} = 3I - A.$$

□